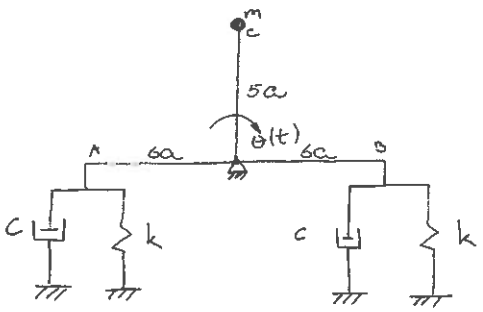
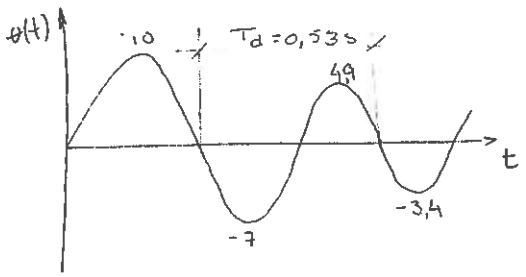
Exame 15/1/2007

1.




$m = 3kg$, massa das barras = 0.

$k = 200\ N/m$

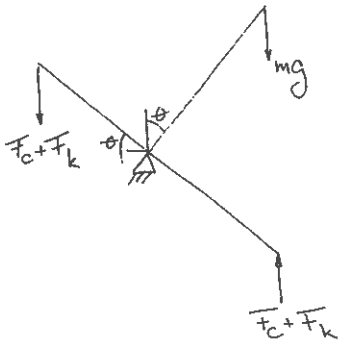



a)

$F_k = k \cdot 6a\theta$

$F_c = c \cdot 6a\dot{\theta}$

$\sum M_o = (J_o^c + J_o^m)\ddot{\theta} = (J_G^m + m(5a)^2)\ddot{\theta}$




$\sum M_o = mg\,5a\sin\theta - 2F_k \cdot 6a\cos\theta - 2F_c \cdot 6a\cos\theta = 25ma^2\ddot{\theta}$ ($J_o \cdot \ddot{\theta}$)

$= \frac{5}{a}mg\theta - 2k(6a)^2\theta - 2 \cdot c(6a)^2\dot{\theta} = 25ma^2\ddot{\theta}$

$25m\ddot{\theta} + (72k - \frac{5mg}{a})\theta + 72c\dot{\theta} = 0$

$25m\ddot{\theta} + 72c\dot{\theta} + (72k - \frac{5mg}{a})\theta = 0$

b) $a = 0,04\ m$

$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{72k - 5mg/a}{25m}} = \sqrt{\frac{72 \cdot 200 - 5 \cdot 3 \cdot 9,81/0,04\,m}{25 \cdot 3}} = 11,96\ rad/s$

c)

$\delta = \ln\frac{x(t_1)}{x(t_2 = t_1 + T)} = \ln\frac{10}{4,9} = 0,71$ ; $\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} = \frac{0,71}{\sqrt{4\pi^2 + 0,71^2}} = 0,113$

d)

$\zeta = \frac{c}{2m\omega_n} \Leftrightarrow c = 0,113 \cdot 2 \cdot 3 \cdot 11,96 = 8,11\ kg/s$

e)

$\omega_d = \omega_n\sqrt{1 - \zeta^2} = 11,88$ ; $T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{11,88} = 0,53\,s$.

f) $\dot{\theta}_0 = 10\ rad/s$ $\quad t \to \theta(t) = 0$

Sistema sub-amortecido tem como resposta

$\theta(t) = A\,e^{-\zeta\omega_n t}\cos(\omega_d t - \phi) = 0,84\,e^{-1,35t}\cos(11,88t - \frac{\pi}{2}) = 0$ → $t = 7,71\,s$

$A_1 = \theta_0 = 0$

$A_2 = \frac{\dot{\theta}_0 + \zeta\omega_n\theta_0}{\omega_n\sqrt{1-\zeta^2}} = \frac{10}{11,96\sqrt{1 - 0,113^2}} = 0,84$

$A = \sqrt{A_1^2 + A_2^2} = A_2 = 0,84$

$\phi = \tan^{-1}\frac{A_2}{A_1} = \frac{\pi}{2}$

h) $\mu < \sqrt{2} \rightarrow \omega = ?$

$$\frac{1}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2+\left(2\xi\left(\frac{\omega}{\omega_n}\right)\right)^2}} \leq \sqrt{2}$$

$$\sqrt{(1-\beta^2)^2+(2\xi\beta)^2} \geq \frac{1}{\sqrt{2}}$$

$$\sqrt{1-2\beta^2+\beta^4+(2\xi)^2\beta^2}$$

$$\sqrt{\beta^4+(4\xi^2-2)\beta^2+1} \geq \frac{1}{\sqrt{2}}$$

$$\beta^4+2(2\xi^2-1)\beta^2+\frac{1}{2} \geq 0 \qquad \text{P/ } \beta>0$$

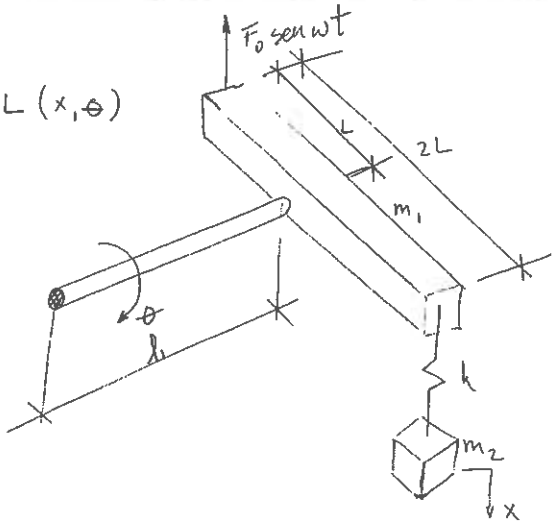
$\beta_1 = 0{,}304 \qquad \omega_1 = 0{,}304 \cdot 11{,}96 = 3{,}63 \text{ rad/s}$

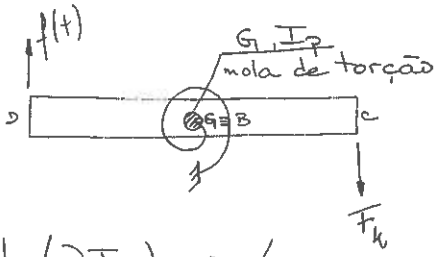
$\beta_2 = 1{,}645 \qquad \omega_2 = 1{,}645 \cdot 11{,}96 = 19{,}67 \text{ rad/s}$

2.

Exame 15/1/2007

GDL $(x, \theta)$




$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) + \frac{\partial T}{\partial q_i}^{\,0} + \frac{\partial V}{\partial q_i} = Q_i$$

$$J_b = \frac{1}{12} m_1 \cdot (2L)^2 = \frac{1}{12} \cdot 3m2^2 = J \cdot m$$

a)

$$T = \frac{1}{2} J_b \dot{\theta}^2 + \frac{1}{2} m_2 \dot{x}^2$$

$$\frac{\partial T}{\partial \dot{q}_x} = m_2 \dot{x} \; ; \; \frac{d}{dt} = m_2 \ddot{x} \qquad ; \qquad \frac{\partial T}{\partial q_\theta} = J_b \dot{\theta} \; ; \; \frac{d}{dt} = J_b \ddot{\theta}$$

$GI_p \frac{d\theta}{dx} = M_t$ como veio de secção cte e $\theta$ é a rotação na extremidade oposta ao encastramento temos:

$$GI_p \frac{\theta}{l_1} = M_t \qquad k = \frac{M_t}{\theta} = \frac{GI_p}{l_1}$$

$$V = \frac{1}{2} GI_p \cdot \frac{1}{l_1} \theta^2 + \frac{1}{2} k (L\theta - x)^2$$

$$\frac{\partial V}{\partial q_\theta} = GI_p \frac{1}{l_1} \theta + Lk(L\theta - x) \qquad ; \qquad \frac{\partial V}{\partial q_x} = -k(L\theta - x)$$

① x: $m_2 \ddot{x} - k(L\theta - x) = 0 \iff m_2 \ddot{x} + kx - Lk\theta = 0$

② $\theta$: $J_b \ddot{\theta} + Lk(L\theta - x) + \frac{GI_p}{l_1}\theta \iff J_b \ddot{\theta} - Lkx + \left(L^2 k + \frac{GI_p}{l_1}\right)\theta = 2L F_0 \operatorname{sen} \omega t$

$$\begin{bmatrix} m_2 & 0 \\ 0 & J_b \end{bmatrix} \begin{vmatrix} \ddot{x} \\ \ddot{\theta} \end{vmatrix} + \begin{bmatrix} k & -Lk \\ -Lk & L^2 k + \frac{GI_p}{l_1} \end{bmatrix} \begin{vmatrix} x \\ \theta \end{vmatrix} = \begin{vmatrix} 0 \\ 2LF_0 \end{vmatrix} \operatorname{sen} \omega t$$

b) $m_1 = m_2 = 3m$ ; $l_1 = L = 1\,m$ ; $k = 160\ N/m$ $GI_p = 80\ Nm^2$

$$\begin{bmatrix} 3m & 0 \\ 0 & m \end{bmatrix} \begin{vmatrix} \ddot{x} \\ \ddot{\theta} \end{vmatrix} + \begin{bmatrix} 160 & -160 \\ -160 & 240 \end{bmatrix} \begin{vmatrix} x \\ \theta \end{vmatrix} = \begin{vmatrix} 0 \\ 2F_0 \end{vmatrix}$$

$$\begin{vmatrix} 160 - \omega^2 3m & -160 \\ -160 & 240 - \omega^2 m \end{vmatrix} = (160 - 3\omega^2 m)(240 - \omega^2 m) - 160^2 = 0 \iff 3m^2\omega^4 - 880\,\omega^2 m + 12800 = 0$$

$$\frac{880m \pm \sqrt{880^2 m^2 - 4 \cdot 12800 \cdot 3m^2}}{6m^2}$$

$$\frac{880/1 \pm 850{,}41/1}{6m^2}$$

$$\begin{vmatrix} \omega_1^2 = \frac{15{,}35}{m}\ (rad/s)^2 \\ \omega_2^2 = \frac{277{,}98}{m}\ (rad/s)^2 \end{vmatrix} \Rightarrow \begin{vmatrix} \omega_1 = \frac{3{,}92}{\sqrt{m}}\ rad/s \\ \omega_2 = \frac{16{,}67}{\sqrt{m}}\ rad/s \end{vmatrix}$$

$$\begin{bmatrix} 360 - 3\omega_i^2 m & -360 \\ -360 & 240 - \omega_i^2 m \end{bmatrix} \begin{Bmatrix} u_{1i} \\ u_{2i} \end{Bmatrix} = 0 \qquad \begin{cases} (360 - 3\cdot\frac{15,35}{m}m)u_{11} - 360u_{21} = 0 \\ -360u_{11} + (240 - \frac{15,35}{m}m)u_{21} = 0 \end{cases}$$

$$160\,u_{11} = 224,65\,u_{21} \quad \Rightarrow \quad \frac{u_{11}}{u_{21}} = 1,40 \quad \Rightarrow \quad \{u\}_1 = \begin{Bmatrix} 1,40 \\ 1 \end{Bmatrix}$$

$$\begin{cases} (360 - 3\cdot 277,98)u_{12} - 360u_{22} = 0 \\ -360u_{12} + (240 - 277,98)u_{22} = 0 \end{cases}$$

$$360u_{12} = -37,98u_{22} \quad \Rightarrow \quad \frac{u_{12}}{u_{22}} = -0,237 \quad \Rightarrow \quad \{u\}_2 = \begin{Bmatrix} -0,237 \\ 1 \end{Bmatrix}$$

$$\{x\} = \begin{Bmatrix} 1,40 \\ 1 \end{Bmatrix} \cos(3,65t - \phi_1)$$

$$\{\phi\} = \begin{Bmatrix} -0,237 \\ 1 \end{Bmatrix} \cos(16,67t - \phi_2)$$

d)

$$\{\phi\}_i = \frac{1}{\sqrt{\{u\}_i^T [m] \{u\}_i}} \{u\}_i$$

$$\{\phi\}_1 = \begin{Bmatrix} 0,534 \\ 0,381 \end{Bmatrix} \quad ; \quad \{\phi\}_2 = \begin{Bmatrix} -0,219 \\ 0,925 \end{Bmatrix}$$

$$[\Phi] = \begin{bmatrix} 0,534 & -0,219 \\ 0,381 & 0,925 \end{bmatrix}$$

$$[I]\{\ddot{\eta}(t)\} + [\Omega^2]\{\eta(t)\} = [\Phi]^T \{f(t)\}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{Bmatrix} + \begin{bmatrix} 15,35 & 0 \\ 0 & 277,98 \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix} = \begin{bmatrix} 0,534 & 0,381 \\ -0,219 & 0,925 \end{bmatrix} \begin{Bmatrix} 0 \\ 2F_0 \end{Bmatrix}$$

$$\begin{cases} \ddot{\eta}_1 + 15,35\eta_1 = 0,381 \cdot 2F_0 = 0,762F_0 \\ \ddot{\eta}_2 + 277,98\eta_2 = 0,925 \cdot 2F_0 = 1,850F_0 \end{cases}$$

e) se $f(t) = F_0 \operatorname{sen}\omega t$ $\quad X(\omega) = ?$ e $\Theta(\omega) = ?$ ; $m = 2kg$ e $\omega = 8 rad/s$

$$[Z] = \begin{bmatrix} 360 - 3\cdot 2\cdot 8^2 & -360 \\ -360 & 240 - 2\cdot 8^2 \end{bmatrix} = \begin{bmatrix} -221 & -160 \\ -160 & 112 \end{bmatrix}$$

$$[Z]\{x\} = \{F\}$$

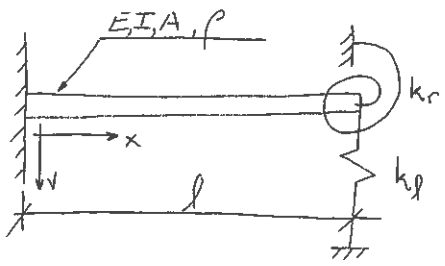
$$\begin{bmatrix} -224 & -160 \\ -160 & 112 \end{bmatrix} \begin{Bmatrix} X \\ \Theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ F_0 \cdot 2 \end{Bmatrix}$$

$$X(\omega) = \frac{112\cdot 0 + 160\cdot 2F_0}{-224\cdot 112 - 160^2} = -6,31\cdot 10^{-3} m = -6,31 mm$$

$$\Theta(\omega) = \frac{-224\cdot 2F_0 + 160\cdot 0}{-224\cdot 112 - 160^2} = 8,838\cdot 10^{-3}$$

3.

Exame 15/1/2007




Condições de Fronteira

$v(0,t)=0$ g

$\frac{\partial v(0,t)}{\partial x}=0$ g

$EI\frac{\partial^3 v}{\partial x^3} = -k_l \cdot v(l,t)$ n

$EI\frac{\partial^2 v}{\partial x^2} = -k_r \frac{\partial v(l,t)}{\partial x}$ n

$v(x,t) = V(x)\, g(t)$

b)

$V(x) = ax^2 + bx + c$

Prob de flexão => 4ª ordem. Tem q ter $V''(x)$

$V''(x) = 2a$ ✓

Verificar cond. geométricas

- $V(0) = 0 \Leftrightarrow a\cdot 0 + b\cdot 0 + c \neq 0$ ✗

$V'(0) = 0 \Leftrightarrow 2a\cdot 0 + b \neq 0$ ✗

Não pode ser aplicável! por isso vamos adoptar como $V(x) = ax^2$

$V(0) = 0 \quad a\,0 = 0$ ✓

$V'(0) = 0 \quad 2a\,0 = 0$ ✓

c)

$$T_{max} = \frac{1}{2}\omega^2 \int_0^l \rho A (ax^2)^2 dx = \frac{1}{2}\omega^2 \rho A a^2 \cdot \frac{l^5}{5}$$

$$V_{max} = \frac{1}{2}\int_0^l EI \left(\frac{d^2(ax^2)}{dx^2}\right) dx + \overbrace{\frac{1}{2}k_l \cdot (ax^2)^2 + \frac{1}{2}k_r \left(\frac{d}{dx}(ax^2)\right)^2}^{\text{para } x = l}$$

- $V_{max} = \frac{2\cancel{a^2} EI}{\cancel{a}} + \frac{1}{2}k_l \cdot a^2 l^4 + \frac{1}{2}k_r \cdot 4 a^2 l^2 = \frac{1}{2}\omega^2 \rho A a^2 \frac{l^5}{5} = T_{máx}$

$\Leftrightarrow \frac{2EI}{a} + k_l l^3 + 4k_r l = \rho \frac{l^5}{5} A \cdot \omega^2$

$$\omega = \sqrt{\frac{2EI/a + k_l l^3 + 4k_r l}{\rho l^5 A \cdot 1/5}}$$

d) i)



ii)



iii)



iv)